17/02/2024

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$\longleftarrow \times \longrightarrow$ B. Sc. Part II (Hons)

3rd Paper.

Vector calculus.

v.v. Impt & difficult sum

I. Prove that

Rough

$$\nabla \times \{(\vec{r} \times \vec{a}) \times \vec{b}\} = -\vec{a} \times \vec{b}$$

where $\vec{r} = (x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$.

Soln.

Here, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\Rightarrow \frac{\partial \vec{r}}{\partial x} = \vec{i}, \quad \frac{\partial \vec{r}}{\partial y} = \vec{j}, \quad \frac{\partial \vec{r}}{\partial z} = \vec{k}$$

Also, $\frac{\partial \vec{a}}{\partial x} = 0 = \frac{\partial \vec{a}}{\partial y} = 0 = \frac{\partial \vec{a}}{\partial z}, \quad \frac{\partial \vec{b}}{\partial x} = \frac{\partial \vec{b}}{\partial y} = \frac{\partial \vec{b}}{\partial z} = 0$

Now $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$

$\therefore$ LHS $= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \{(\vec{r} \times \vec{a}) \times \vec{b}\}$

$= \vec{i} \times \frac{\partial}{\partial x}\{(\vec{r} \times \vec{a}) \times \vec{b}\} + \vec{j} \times \frac{\partial}{\partial y}\{(\vec{r} \times \vec{a}) \times \vec{b}\}$

$+ \vec{k} \times \frac{\partial}{\partial z}\{(\vec{r} \times \vec{a}) \times \vec{b}\}$

$= \vec{i} \times \left[\left(\frac{\partial \vec{r}}{\partial x} \times \vec{a}\right) \times \vec{b} + \left(\vec{r} \times \frac{\partial \vec{a}}{\partial x}\right) \times \vec{b}\right.$

$\left. + (\vec{r} \times \vec{a}) \times \frac{\partial \vec{b}}{\partial x}\right]$

$+ \vec{j} \times \left[\left(\frac{\partial \vec{r}}{\partial y} \times \vec{a}\right) \times \vec{b} + \left(\vec{r} \times \frac{\partial \vec{a}}{\partial y}\right) \times \vec{b} + (\vec{r} \times \vec{a}) \times \frac{\partial \vec{b}}{\partial y}\right]$

$+ \vec{k} \times \left[\left(\frac{\partial \vec{r}}{\partial z} \times \vec{a}\right) \times \vec{b} + \left(\vec{r} \times \frac{\partial \vec{a}}{\partial z}\right) \times \vec{b} + (\vec{r} \times \vec{a}) \times \frac{\partial \vec{b}}{\partial z}\right]$

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using the values from (i) & (ii) in LHS, we get

$$LHS = \vec{i} \times \left[(\vec{i} \times \vec{a}) \times \vec{b} + \vec{0} + \vec{0} \right]$$

$$+ \vec{j} \times \left[(\vec{j} \times \vec{a}) \times \vec{b} + \vec{0} + \vec{0} \right]$$

$$+ \vec{k} \times \left[(\vec{k} \times \vec{a}) \times \vec{b} + \vec{0} + \vec{0} \right]$$

$$= \vec{i} \times \left[(\vec{i} \times \vec{a}) \times \vec{b} \right] + \vec{j} \times \left[(\vec{j} \times \vec{a}) \times \vec{b} \right]$$

$$+ \vec{k} \times \left[(\vec{k} \times \vec{a}) \times \vec{b} \right]$$

$$= \vec{i} \times \left[(\vec{i} \cdot \vec{b}) \vec{a} - (\vec{a} \cdot \vec{b}) \vec{i} \right]$$

$$+ \vec{j} \times \left[(\vec{j} \cdot \vec{b}) \vec{a} - (\vec{a} \cdot \vec{b}) \vec{j} \right]$$

$$+ \vec{k} \times \left[(\vec{k} \cdot \vec{b}) \vec{a} - (\vec{a} \cdot \vec{b}) \vec{k} \right]$$

$$= (\vec{i} \times \vec{a})(\vec{i} \cdot \vec{b}) - (\vec{i} \times \vec{i})(\vec{a} \cdot \vec{b})$$

$$+ (\vec{j} \times \vec{a})(\vec{j} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})(\vec{j} \times \vec{j})$$

$$+ (\vec{k} \times \vec{a})(\vec{k} \cdot \vec{b}) - (\vec{k} \times \vec{k})(\vec{a} \cdot \vec{b})$$

$$\boxed{\text{using } (\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{b} \cdot \vec{c}) \vec{a}}$$

Now we $\vec{i} \times \vec{i} = \vec{0}$, $\vec{k} \times \vec{k} = \vec{0}$
$\vec{j} \times \vec{j} = \vec{0}$,

$\therefore$ LHS $= (\vec{i} \times \vec{a})(\vec{i} \cdot \vec{b})$

$+ (\vec{j} \times \vec{a})(\vec{j} \cdot \vec{b})$

$+ (\vec{k} \times \vec{a})(\vec{k} \cdot \vec{b})$

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

$\therefore \vec{i} \times \vec{a} = \vec{i} \times (a_1\vec{i} + a_2\vec{j} + a_3\vec{k})$

$= a_2\vec{k} - a_3\vec{j}$

$\vec{j} \times \vec{a} = \vec{j} \times (a_1\vec{i} + a_2\vec{j} + a_3\vec{k})$

$= a_3\vec{i} - a_1\vec{k}$

$\vec{k} \times \vec{a} = \vec{k} \times (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) = a_1\vec{j} - a_2\vec{i}$

Also $\vec{i} \cdot \vec{b} = \vec{i} \cdot (b_1\vec{i} + b_2\vec{j} + b_3\vec{k}) = b_1$

$\vec{j} \cdot \vec{b} = \vec{j} \cdot (b_1\vec{i} + b_2\vec{j} + b_3\vec{k}) = b_2$

$\vec{k} \cdot \vec{b} = \vec{k} \cdot (b_1\vec{i} + b_2\vec{j} + b_3\vec{k}) = b_3$

using these values in LHS, we get

LHS $= b_1(a_2\vec{k} - a_3\vec{j}) + b_2(a_3\vec{i} - a_1\vec{k})$

$+ b_3(a_1\vec{j} - a_2\vec{i})$

$= (b_2a_3 - b_3a_2)\vec{i} - (b_1a_3 - b_3a_1)\vec{j}$

$+ (a_2b_1 - a_1b_2)\vec{k}$

$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$

= RHS Proved